Non-Euclidean Space: Is the Universe Round?

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by

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Our topic is non-Euclidean Geometry in three dimensions, not as an idle amusement, but as a likely reality. This is not to say that you will not be amused -- I certainly hope you will be.

The goal is to develop a qualitative understanding of the strange and wonderful possibilities that exist in theory -- and perhaps in fact. The approach will be exploratory, that is, we will consider explorations which are conceivable if not practical. We will consider not only explorations of three-space by human cosmonauts, but also analogous explorations of two-space by "flatlanders," the famous 2-D people invented by "A. Square," aka Edwin A. Abbott. The motivation here is to help overcome our prejudices regarding the character of three-space by seeing how the (unjustified) preconceptions of the flatlanders are easily understood by clever people like us who enjoy a life in three dimensions. This, it may be hoped, will encourage a more open and perhaps more humble attitude in dealing with our own instinctive preconceptions regarding the possible character of the space in which we exist.

Our first goal is to get a feel for, if not to understand, an object called the three-sphere. By this we mean a three dimensional universe which is finite in volume yet has no boundaries. It seems safe to say that this thing can't be visualized, but there is no reason why it couldn't be explored if it were small enough in relation to the product of our speed and our longevity.

We will begin by eavesdropping on the analogous exploration carried out by intrepid flatlanders who, unknown to themselves, live in a two-sphere. By a two-sphere I mean the surface of a perfectly round ball. Some people will think of this as a three dimensional figure because it’s possible to represent the two-sphere as an object imbedded in three-space, just as all of us humans envision it. But this embedding is not a mathematical necessity. In fact, as we shall see later, if three-spheres exist, then two-spheres will exist in them which will appear to be perfectly flat! But that's for later on. We shall imagine that the flatlanders are tiny in the sense that their dimensions and that of their houses, etc., are very small in comparison to the circumference of the sphere in which they live. This being the case, we may assume that their minds have evolved to conceive of their world as being flat, a realm in which Euclidean geometry is exact, not just a fair approximation for small figures. For whatever reason, plain curiosity perhaps, they decide to explore their space. Thus they send out some number, N, of explorers from a well-marked point, which we will call home.

Each flatlander is to move in (what they think of as) a straight line a certain exact distance each day. Also, the angle between adjacent paths is to be the same all around: $(2\pi/N)$. Each day they will measure the distance, d, between adjacent observers as well as the angle, a, between the next guy and the path back home. At first they will detect no departures between the measured values and their Euclidean predictions:

$$d = 2L\sin(\pi/N), \qquad a = \pi/2 - \pi/N$$

But further on they will be able to detect increasing discrepancies between measured and predicted values. For as we can plainly see, they are not on a plane, but on a sphere and we can predict that, after traveling a distance, L, they will measure

$$d = 2R\ \arcsin[\sin(L/R)\sin(\pi/N)], \qquad a = \mathrm{arcctn}[\cos(L/R)\tan(\pi/N)].$$

(You need pay no attention to these equations -- you can <u>see</u> what happens!) The distance between adjacent explorers does not increase as fast as it would if their world were actually flat (a plane), and the angle between home and the next guy is not constant, but begins to increase. When they are very near what we might call the equator -- the great circle halfway between home and its antipode -- the distance between adjacent explorers doesn't increase at all, and the angle is ninety degrees. This is incomprehensible (or at least unvisualizeable), to the flatlanders, even as it is obvious to us three-D types. As they proceed, the situation becomes even more amazing to the intrepid explorers: For now their adjacent fellows seem to be leading, that is, to be slightly in front of the ninety degree line! And worse, their distance begins to lessen! What amazement when finally they all converge at a single point! Of course to us there is no surprise: they have merely

reached that unique point that is as far away from home as is possible, the antipodal point.
Now we turn to our own explorations. Let's say we send out twelve space ships in directions such that each moves at right angles to one of the twelve facets of a regular dodecahedron. At first things go as we "flatspacers" would presume, but as the ships go on, discrepancies appear: The distance between a spacecraft and its five nearest neighbors does increase, but not in strict proportion to the distance from home. Also the angle between neighbors and the road home is not constant, but increases. There comes a time, in fact, when the distance between the spaceships doesn't increase at all as they move outward, and at this time the angle to adjacent ships becomes ninety degrees! At this stage the spaceships lie on what is logically called a great sphere, a 3-D analog of a great circle on a two-sphere. And this sphere is flat, as far as any of the cosmonauts can ascertain!

Subsequently as the ships move onward, to the amazement of all the cosmonauts, the distance between ships begins to decrease and neighbors seem to be leading. Finally, all of the space ships converge on a single point. They have arrived at that point which is as far from home as one can be: the antipodal point.

Reflecting on the mystification suffered by the flatlanders, we are now put in the circumstance of having to truly sympathize with them, for it is little comfort to be told that some four dimensional critter can easily understand how our space closes on itself. It's curved, the critter claims, but the center of curvature lies outside of our three space, in a direction which is at right angles to any direction in which we can point.

Incidentally, the three-sphere is a very special case: it's very symmetric. For the general non-Euclidean three-space, not four but <u>six</u> dimensions are required to carry out the representation of the three-space as a curved figure imbedded in the Euclidean higher-dimensional space.

Let's pause here to consider a couple of implications of such a global geometry. Consider the radiation produced by a star or a galaxy. The intensity of light from such a source decreases as the light proceeds outward because the energy emitted is conserved and lies on a sphere whose surface area increases as it moves outward. But in a three-sphere, as we have seen, the area of a outward moving sphere doesn't always increase. Once it has reached the great sphere halfway to the antipode of the source, the area of the sphere begins to decrease, and thus the intensity would thereafter increase as the light moved onward. It would, in fact, collapse on the antipode, creating there a "ghost" star or galaxy; or would do so, were it not for imperfections of the optical properties of the universe, and limitations of the time available from the onset of the "big bang."

Another surprising feature of the three-sphere is the number and character of the ways in which it can be subdivided into exactly congruent sub regions. Again it helps to first think of the two dimensional case. Each of the five so-called regular solids – the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron – corresponds to a way of dividing the two-sphere into congruent sub regions. Just center the solid inside the sphere and then project the edges onto the surface of the sphere.

Notice that three of the tilings, tessellations in mathspeak, employ triangular sub regions. In tiling the plane with triangles, one must have six converging on each vertex, and one can make the triangles any size so long as they're all the same. But as you know, the angles of a spherical triangle add up to more than 1 80 degrees, and the bigger the triangle, the greater the excess. Thus six equilateral triangles of any finite size cannot be brought together at a point on a sphere. But five triangles, if they are just the right size in relation to the circumference of the sphere, can be brought together in a tight fit. One then just continues to add triangles of the same size, and the result is a tessellation comprising twenty triangles, corresponding to the icosahedron. Or one may choose to bring only four triangles together at a vertex: the triangles must be larger, and in this case one finds the whole sphere covered with only eight triangles - the octahedral tessellation. Finally, one can bring just three triangles together at a vertex. This results in just four sub-regions: the tetrahedral division of the two-sphere.

On a flat plane, one brings four squares together to tile the surface. On the sphere, bringing three squares together at a vertex results in a six sub region division for the sphere, a tessellation generated by the cube. Finally, one may bring together three pentagons together at a vertex. On a flat plane, they would rattle about: but on the sphere their angles grow fatter as they grow in size in relation to the circumference of the sphere, and at some size they fit together perfectly. Twelve such pentagons fill up the sphere - the dodecahedral tessellation.

In cutting up a three-sphere, one has to use the whole volume of the regular solids as the sub regions. Also the same regular solids must be used to specify how the corners of the volumes are to meet at the vertices. What we do in constructing these tessellations is to think of bringing together corners that would "rattle around" in flat (Euclidean) space. For instance, we know that eight cubes can be brought together at a vertex in flat space. Each cube has a three facets coming together at each of its corners, so the eight cubes meet in an octahedral corner – a figure having eight triangular facets. Now in a three-sphere we must bring together fewer cubes. Since they have "triangular" corners, we must use a vertex model that also has triangular facets. The only vertex model with fewer than eight facets is the tetrahedron. This results in the (unjustly) famous "tesseract," usually described as "a regular four-dimensional solid." I feel that it is much better thought of as a regular tessellation of the three-sphere. Anyway there are eight sub regions (volumes); 24 boundary surfaces, each of which is a section of a great sphere (which would appear to be perfectly flat to inhabitants); 32 edges, all geodesics (seeming straight lines to the natives); and finally 16 vertices.

Perhaps the most fundamental tessellation of the three-sphere is that in which four tetrahedra meet in at a tetrahedral vertex. It may surprise you to learn that this results in just five volumes, ten bounding surfaces, ten edges and five vertices. This is the "simplex" in four dimensions, and its generalization exists in any number of dimensions. (It is preceded by the line segment, the triangle, and the tetrahedron, in one, two, and three dimensions, respectively.)

I think the most charming tessellation is that in which one brings together four dodecahedrons at each tetrahedral vertex. This results in no less than 120 volumes, each a dodecahedron, 720 pentagonal bounding surfaces, 1200 edges, and 600 vertices. This tessellation, as do all others, has a conjugate tessellation, which is derived by interchanging points and volumes, on the one hand, and edges and surfaces, on the other. In the present case this results in a tessellation in which twenty tetrahedra meet at each icosahedral vertex, yielding 600 volumes, 1200 surfaces, 720 edges, and 1 20 vertices.

Altogether there are six regular tesselations of the three-sphere (a.k.a. regular four dimensional solids). Besides those already mentioned there is the one conjugate to the "tesseract" in which eight tetrahedra meet in octahedral vertices, (16 volumes, 32 surfaces, 24 edges, 8 vertices), and the self-conjugate tessellation in which six octahedra meet at cubical vertices, (24 volumes, 96 surfaces, 96 edges, 24 vertices). The simplex is also self-conjugate, of course.

Just imagine the incredible generalizations of the game of chess that one could devise using these three dimensional "hyper-boards!"

Before continuing with our investigation of the three-sphere, let's consider another type of non-Euclidean geometry, namely that which appears in connection with the general theory of relativity. We will consider then the geometry in the vicinity of a massive body.

One preliminary consideration must first be dealt with. Obviously, geometry involves the interplay of inherent geometric structure and some system of measurement.

When my son was a little boy, he and I used to enjoy tall tales which we would invent. One of mine concerned a mysterious swamp which lay a mile or so from my boyhood home. The thing was, I told my son, you could walk around the swamp in perhaps two hours, but it seemed to be impossible to cross directly. I myself, I told him, have walked steadily in a straight line for a whole day without emerging or even seeing the far side! How can that be, he wondered. Well, I don't know I said, but my guess is that as I proceeded into the swamp, I began to shrink, so that my steps got smaller and smaller. Not that I could really notice though, because everything looked normal. So if I was shrinking, so was everything else -- the trees and plants, bird and snakes – everything. The other strange thing was that people outside didn't agree with me as to the time I'd been gone. My whole day was to them half an hour! So my watch must have speeded up as I shrank.

At the time I was telling this story, I had no idea that there is a fantastic parallel between this story and a situation that naturally arises in the general theory of relativity. In the field of a gravitating body, a mass falling from rest acquires kinetic energy. Experiments show that this energy is exactly equal to the work which would have to be done to restore the body to its original position. To incorporate this fact, the concept of potential energy was invented. When, say, one struggled hauling a stone up the side of a pyramid, that one was doing work was obvious, and the theory said that this work was going into potential energy. No one, however, was specific as to where or in what form this potential energy existed. Incredibly, even after Einstein wrote $E = mc^2$, no one, as far as I know, identified the hiding place of this theoretical potential energy. It turns out, at least in my view, that the potential energy is literally embodied as an increase, usually very tiny, in the rest mass of the object being raised.

If this is accepted, and if one knows a few fundamental facts in the quantum theory, one is able to understand an effect, predicted by Einstein and confirmed by very precise experiments, known as the time dilation effect. It is also known as the gravitational red shift, but a much better name would be "the gravitational slowing of clocks."

But this is not all. If one identifies potential energy with a change in rest mass, one will also predict the gravitational dilation of measuring rods! Both of these effects are simple straightforward consequences of the dependence of rest mass upon position in a gravitational field.

I must tell you that as of now the physics community has not endorsed this proposal. But it certainly will, and soon.

However this may turn out, the point I want to make here is that in describing the geometry of the regions near a gravitating body, we are forced to choose between alternative schemes for distance measurement. The choice will have a dramatic effect upon the perception we have of the inherent geometry of the region near a star or a black hole.

The choice we shall make is not the usual one, which is to rely upon local measurements made with elongated measuring rods. Instead, we will use a system of echo-ranging – radar if you will – using not the local tardy clocks which are slowed by the effects of gravitation, but distant clocks which, in a time-independent situation, may be relied upon to provide an invariant frequency standard. This will in effect provide us with measuring rods that do not elongate in a gravitational field.

We are going to return to the technique of the imagined exploration in order to explain the geometry in the neighborhood of a gravitating body such as a star or a black hole. Again we will use the flatlanders to help us overcome the prejudices that our hard-wired Euclidean brains force upon us. I want you to imagine that the flatlanders inhabit a surface which at great distances conforms to a planar, that is to say, flat surface, but which, in the region of interest, sweeps upward like the base of a round volcano. To stretch your imagination even further, I want you to accept that surface as it sweeps upward goes to the vertical and beyond, forming an expanding upper surface over the original one, like the cap of a morel mushroom. This is an upside-down version of the diagram that is often used to illustrate the so-called "wormhole" beloved to expositors of relativity, myself included. I want to add one more crazy feature to the structure. At some stage, as we follow the upper surface upward and outward, it becomes horizontal and beyond that it develops the shape of a floppy hat brim because the perimeters of circles increase faster than the distance between them.

With this picture in mind we now imagine our intrepid flatlanders disposed in a circle surrounding the region of interest. Their expectation as they move toward one another is that the distance between adjacent explorers will decrease in strict proportion to the distance advanced: $\Delta C = 2\pi\Delta R$. But of course we can see that this expectation will not be realized. As they move inward they are (as we can see and they cannot) carried upward and consequently they do not close as fast as they had expected. This is only mildly surprising to them, but as they continue beyond what we see to be the vertical section, they are astounded to discover that they are clearly moving apart as they continue what they think of as their inward marching. However amazing, it is clear to the explorers that they are now marching away from one another. When they reach the "floppy hat" region the distance between adjacent explorers increases even faster than it would if they were on a flat surface marching outward. Somehow, magically, a whole new universe, roomier than the old familiar one, has been discovered lying within a circle of finite radius!

So now we shift to our own three dimensional world to follow twelve cosmonauts, maybe better called astronauts in this context, in their investigation of the geometry of a black hole. For this exploration they are piloting spacecraft fitted with powerful rocket thrusters to enable them to move or remain motionless as they choose. Here the experience of the astronauts parallels exactly that of the flatlanders just outlined. Starting from positions disposed on the facets of a dodecahedron surrounding the black hole, they maneuver directly inward together, all at the same rate. At first each draws nearer to his five neighbors, but at a rate which decreases and departs ever more from that expected as they continue inward. At one stage, which may be called the throat, the closing ceases altogether and thereafter reverses, so that adjacent astronauts are now clearly drawing away from one another even as they continue to move in the direction they had identified as inward! Clearly, it is reasonable to conclude that they are in fact moving outward into a second universe which somehow was, and is, hidden inside the finite sphere containing the black hole. The situation is symmetrical in the sense that, looking back, the astronauts see that their old familiar infinite universe is contained inside a finite sphere lying in the new universe.

This weird geometry explains the peculiar behavior of centrifugal force in the field of a black hole. Let's send one spacecraft back to the starting point, and have the pilot maneuver his craft in a circular orbit about the black hole. What we want to do is to take note of, in a qualitative way, the amount of thrust required to maintain the orbit.

In the outer regions, things go as expected: at low orbital speeds, the rocket thrust must directed outward to overcome the difference between the force of gravity and the centrifugal force, which is directed outward. At just the right orbital speed, no thrust is required: the centrifugal force is equal and opposite the force of gravity. At higher speeds the rocket thrust must be directed inward because the outward centrifugal force exceeds the force of gravity.

But at the throat a very peculiar thing is observed. Here the thrust required to maintain an orbit is independent of the orbital speed! This can only mean that the centrifugal force has vanished. And this can happen only if the path of the orbit is a geodesic. Thus the sphere at the throat appears to the astronauts to be flat! And in fact light rays directed along the surface of this sphere will (unstably) orbit on it.

"Inside" the throat, one discovers that the "centrifugal" force is inaptly named for now it acts inwardly, as inward was originally defined. Thus in order to maintain a circular orbit inside the throat, the direction of thrust must always be directed outward (in the original sense), regardless of the orbital speed, and the amount of thrust required increases as the orbital speed increases.

Here I must admit to having glossed over a complication. This has to do with the previously discussed length dilation effects. In the above "thought" exploration, measurements were assumed to be made using radar techniques based upon a distant clock. Everything I said is true, but I failed to mention that things would look very different to the astronauts themselves, for they themselves, as well as their spacecraft, would swell as a result of the reduction of the proper masses of their constituent atoms. Their perception of distance would be markedly different, and they would not agree that they were moving apart as they moved inward inside the throat. In fact they would, because of their bloating, eventually crash into one another. The facts regarding "centrifugal force," the amount and direction of thrust, however, remain unchanged. In my view, this makes the case for accepting the validity of the geometry as described by the radar & distant clock technique.

So now we have seen another example of non-Euclidean geometry, which is arguably more amazing than our first example, the three-sphere. Here we discover that two universes, each infinite, can coexist and even be connected by a sphere which appears to be flat!

Rudy Rucker, a brilliant popularizer of science and mathematics, suggests in his book, The Fourth Dimension, a way of thinking that seems to help some people feel more comfortable with a geometry that incorporates more than one universe. He suggests that we accept as real a fourth spatial dimension. This is not to be confused with the idea of the fourth dimensionality of relativity, which merely characterizes time as a fourth dimension in order to indicate that time-like and space-like measure become interconnected when comparisons are made between systems in relative motion. Rucker's idea implies, among other things, that by moving in the fourth dimension access to the interior of a bank vault would be possible without penetrating the walls of the vault. Again, we reason by analogy to a flatland situation. A bank vault in flatland would look to us like an enclosure drawn on a sheet of paper. We could not only see the interior, but could reach right in and remove anything without going through the "walls" of the vault.

In exactly the same way, a critter existing in four dimensions, three of ours plus one not accessible to us, could look right at any point inside our bodies, and could remove, modify, or add structures without recourse to a scalpel. He would see our whole three dimensional world as a three dimensional surface!

Returning to the two universe problem, our four dimensional friend would see the two universes as being parallel to one another, connected by a tunnel-like structure (wormhole), but elsewhere separated by a distance purely in the fourth dimension. In fact, her picture would be perfectly analogous to that we used to introduce the flatlander's exploration of a two-dimensional black hole: the volcano/morel mushroom/floppy hat picture.

An interesting variation of the geometry just described is that which appears in the description of a rotating black hole (the Kerr solution to Einstein's equations). In this case the two universes are connected not by a spherical surface but by a disc surrounded by a ring singularity. As soon as one crosses the disc from above, a second universe is entered. If one moves in a circle which encloses the singular ring, one does not return to the original location. In order to return to the original position in the original universe, one must go twice around the circle.

This strange way of connecting two three-spaces is intriguing to me because it seems possible to invoke this structure as the basis of the so-called string theory of elementary particles: the string being identified with the ring singularity, so that particles are little navels connecting the outer universe with an inner universe. Having said this, honesty demands that I frankly admit to monumental naiveté and near perfect ignorance in these matters. And so we will drop the matter right here.

Returning now to the realm of the three-sphere, I would like to introduce you to a most amazing mental construct. This is what I think of as the great torus in the three-sphere. This is a surface lying in the three-sphere which has the nature, that is to say the topology, of a torus, which is a mathspeak for doughnut. The first amazing thing is that no point on this torus is distinctive. This is in contrast to the doughnut of ordinary experience, where points on the inner side of the "hole" lie on a surface that is saddle-shaped in contrast to points lying on the outside of the doughnut. The second amazing thing is that this torus divides the entire three-sphere into two separate but interlocked sub regions.

Let's see how this structure may be inferred. We begin by imagining a geodesic line. Such a line seems to us to be perfectly straight in that the shortest path joining any two points on the line coincides exactly with the line itself. (Of course, in the view of some superior six-dimensional critter, we may well be deceived regarding its straightness.) As we have previously argued, this line will close on itself, forming a great circle. Now through a point on this geodesic, we construct a surface by rotating a second geodesic line about the point, always keeping it at right angles to the first. Since it was generated by a geodesic, this surface will in fact be a great sphere. Think of the first line as being vertical and the surface as being horizontal. Now halfway between the original point and its antipodal point, the horizontal surface is going to intersect the great sphere centered on the original point. The intersection of two great spheres will always result in a great circle, that is, in a geodesic. Thus we have constructed a horizontal geodesic which encircles our original vertical geodesic. But certainly the vertical geodesic must encircle the horizontal one since it passes through the antipodal point which lies halfway around the whole space beyond the horizontal geodesic. The two geodesics are thus mutually encircling! Finally we construct the great torus itself, as the surface consisting of points which lie halfway between the two geodesics. A vertical section of this surface will be a lesser circle centered on the horizontal geodesic, while a horizontal section will be a lesser circle centered on the original vertical geodesic.

Interestingly, diagonal sections (at ± 45 degrees) of the great torus are themselves geodesics. From this comes another way of imagining how a great torus may be generated. Imagine a tight rope walker moving along one of these diagonal geodesics using as a balance pole another geodesic at right angles to the first. As the tight rope walker moves, he slowly rotates the "balance pole" geodesic at just the rate so as to complete one revolution in traversing the entire circumference of the "rope" geodesic.

Now this whole exercise may seem to be without purpose except perhaps entertainment for the easily amused. But I think it may be relevant to a really mysterious characteristic of our universe. This is the remarkable fact that nature does not deal even handedly with left and right, at least when the so-called weak force (whereby, for instance, neutrons decay into protons, electrons and antineutrinos) is involved. Left-handed and right-handed spins are not merely mirror images of one another.

Long time victims of my talks will recall that I feel strongly that the aether is a respectable concept in physics, and that accounting for the effects of the motion of the aether is vital in understanding the dynamics of the rotating black hole (Kerr solution). Let me very briefly describe the situation. Imagine a set of mirrors set up in the equatorial plane so that light signals, bouncing from mirror to mirror, may be sent around the black hole. One finds that signals sent in the direction of rotation require less time to complete the circuit than do ones directed oppositely. And the difference can be accounted for exactly if one assumes that there is an aether and that it is in motion about the black hole. The same assumption also affords a simple explanation of the otherwise arcane photon orbits that are predicted for a rotating black hole. So I ask you to accept the idea of not just an aether, but a dynamical aether capable of hydrodynamic-like behavior.

Armed as we are now with the picture of intertwined geodesics and the great torus, let us now consider aether flows in a three-sphere. First think of steady circular flow around the vertical geodesic. Progressing outward the circular flow lines will grow larger and larger until, halfway to the antipode, they will reach maximum size, and will in fact be coincident with the horizontal geodesic. Thus flow <u>along</u> one geodesic is naturally associated with circular flow <u>around</u> the conjugate geodesic. Suppose now that one superimposes steady flows along both geodesics. The flow along the vertical geodesic will induce a rotational flow around the horizontal geodesic, and the flow along the horizontal geodesic will induce a rotary flow about the vertical. Thus one would see everywhere a vortex flow, with flow lines in the shape of the screw thread, which would be everywhere either right-handed, or everywhere left-handed. Space itself would be everywhere spinning. If one imagines that it's improbable that the aether would just happen to be motionless, then it would seem natural that the physics of the universe would not be indifferent to the handedness of particles which are themselves spinning. In fact, the famous experiment of Lee and Yang (19727) showed that is just the case with the actual universe (at least in the tiny region that we occupy).