Pythagoras, The Music of the Spheres, and the Wolf Interval*

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1 Introduction

There is geometry in the humming of the strings
... there is music in the spacing of the spheres

Although most of his accomplishments are well-known, Pythagoras published nothing that has survived. He lived ca. 570 to ca. 490 BCE[1] and was controversial in his day. His accomplishments are known to us by the writings of historians and his followers. Iamblichus[2] 245-325 BCE, wrote extensively of him. It cannot even be established that the accomplishments attributed to him were the works of one person or of many, spread in time across several generations. Was he one person, or a fusion, a meld of several people? We may never know. Though I refer to him as one distinct person, I really am referring to that meld.

He was the first mathematical physicist, or theoretical physicist, in that he attempted to explain the cosmos in terms of his wave theory of the string ("string theory!"), and not by relying upon supernatural causes. This theory came to be called the *Music of the Spheres*.

A central question I have about Pythagoras is, why did he become or choose to become a cult figure? Cults were popular in 6$^{\text{th}}$ centure BCE, and there were many diverse cults based on everything from the worship of bronzed pets to etherial abstractions. Heraclitus, a contemporary of Pythagoras, based his view of All on

[1]Huffman, Carl, *Pythagoras*, The Stanford Encyclopedia of Philosophy (Fall 2011 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2011/entries/pythagoras/>.

[2]Iamblichus, 1991, *On the Pythagorean Way of Life*, John Dillon and Jackson Hershbell (trans.), Atlanta: Scholars Press.

the duality that “Change is the only constant.”Pythagoras, by way of his interest in mathematics, posited that “All is Number.”By number, he meant the counting numbers, or positive integers. He did not include zero or negative numbers. For everything else that didn’t fit comfortably into the number rubric, he claimed that the universe consists of “The Limiting and the Unlimited.”

He was a talented and cosmopolitan man, being an accomplished lyre player, and studied mathematics with Thales and mystery religions in Egypt. He started a cult when cults were cool. At his academy, initiates had to undergo a long apprenticeship. There were dietary restrictions: vegetarianism without beans and no animal flesh of any kind. Silence was also considered a virtue, as well as personal cleanliness, utter loyalty, and pure linen clothing[3].

One of the few known facts is that the local people in Croton, Italy, where he established his academy, turned against him and destroyed his academy. It is thought that Pythagoras escaped.

The consensus seems to be that his academy got involved in local politics and that he alienated the local residents. There is no definitive record, however. Perhaps the locals thought of him as a devil or demon. This would be consistent with the belief that he had magical powers, but it is likely a cop out. The inhabitants of the Mediterranean were tolerant of mystery cults. Here, permit me an after dinner speculation: perhaps, and more intriguingly, some of his disciples turned against him because there were subtle flaws in his reasoning of some of his claims. This, together with the grievances of the locals, may have brought about his downfall.

The Pythagorean Theorem is above reproach, but there is some controversy about two other important mathematical inventions (or discoveries) associated with him.

2 The Pythagorean Theorem

Pythagoras’ crowning achievement is the theorem named after him. It is not clear, however, that his proof was the first or that it was original with him. The ancient Babylonians may have first proved it. Strikingly, the proof of the Pythagorean Theorem is quite simple. When I was a student, none of my geometry or algebra texts, or teachers, for that matter, ever demonstrated a proof. How many of you can prove it? There are now hundreds of proofs of the theorem, including an early one by Euclid. But Euclid’s proof is overly complex, and consists of dozens of steps. The simplest and most elegant proof is a two-line exercise in algebra, and a two figure construction in geometry. I record it here for the sake of all posterity who will also likely be vexed by absent-minded geometry teachers. Here is a description of the problem. See Figures 1 and 2:

[3]Paul Calter, *Pythagoras and the Music of the Spheres*, ©1998, Dartmouth College

Figure 1: The Pythagorean Theorem

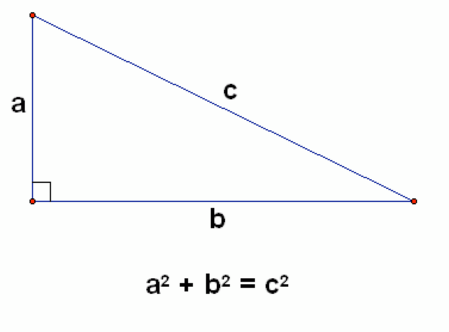
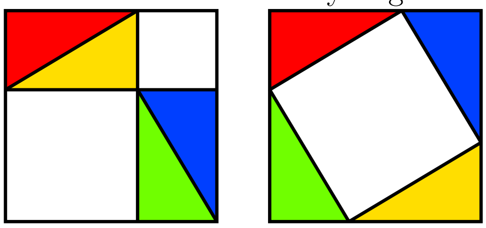


Figure 2: Pictoral Proof of the Pythagorean Theorem



The algebra follows. The area of the large square in Figure 2 is $(a+b)^2$. The area of each triangle is $\frac{1}{2}ab$, and the area of the small square is c^2. Thus we see that

$$(a+b)^2 = c^2 + 4\left(\tfrac{1}{2}ab\right) \rightarrow a^2 + b^2 = c^2$$

3 Irrational Numbers

Pythagoras probably invented rational numbers. A rational number r is a number that can be written as the ratio of two integers: $r = \frac{p}{q}$, with $q \neq 0$.

The first controversy concerns irrational numbers. What are irrational numbers and why might they be so wicked? There are infinitely many of them, and they *cannot* be represented as a ratio of two integers. They are densely packed into the number line (any subinterval of the number line contains an infinite number of irrational numbers). Here is the example that was thought to have confounded Pythagoras: Consider b the square root of 2:

$$b = \sqrt{2},$$

is there any rational number that satisfies

$$b^2 = 2?$$

If b is rational, then $b = \frac{p}{q}$, where p, q are integers. This requires that $p^2 = 2q^2$, and we easily see that there are no integers or whole numbers that can satisfy this relation, because a perfect square would have to contain only *even* powers of integers.

Pythagoras was first skeptical of the existence of irrational numbers because this contradicted his belief that "All is Number". Thus Pythagoras' initial denial may have disillusioned some of his more astute followers and resulted in either defections or a conspiracy to cover it up. Eventually, Pythagoras did have to acknowledge the existence of irrational numbers. He may have been the author of the proof, which is short and elegant. It may be the first example of a *proof by contradiction* (reduction to the absurd) that is a standard item in the mathematician's toolbox. His followers may have viewed his reversal as a fall from grace. He was damned if he did, and damned if he didn't. Did this sort of failing warrant a torches and pitchforks remedy? No, but there is more.

4 Harmonics of the String

His investigations into the harmonics of the string probably resulted in the invention of all the greek musical scales: the *tonic, dorian, phrygian, lydian, mixolydian* being the first five. As I shall show, he discovered an early wave theory. In any case, the historical, or rather, archaeological record, is so incomplete that no definite conclusion can be reached.

He was reputed to be an expert lyre player, so he must have been familiar with the harmonics of the open string. Please refer to Table 1. For instance, if a taut string is merely touched at the center, so that the ratio of subdivided intervals is exactly 1:1, the string will emit a note that is an octave higher than the fundamental of the string. The note of the stopped string is also an octave higher. The stopped note is in a ratio of 1:2 in wavelength to to open string. Thus its frequency will be twice as high, because its wavelength is twice as short. Here we see that Pythagoras was developing the beginnings of a wave theory. Even the frequency and speed of light is governed by a reciprocal relationship. Let f is the frequency, ν the wavelength, and c the speed of light in the vacuum:

$$f = \frac{c}{\nu}.$$

If the string is touched (but not stopped) in the ratio of 1:2, the sound emitted is that of the fifth interval higher than the fundamental. For example, if the

Table 1: Pythagoras' "String Theory "

Fret on the Guitar	Ratio of Harmonic	Note Heard on the 6$^{\text{th}}$ String, cycles per sec.	Ratio of Stopped : Unstopped	Note Heard cycles per sec.
open				E 330
5	1:3	E 1320	3:4	A 440
7	1:2	B 990	2:3	B495
12	1:1	E660	1:2	E 660

fundamental of the string is E (330 cycles/sec), then the sound of the harmonic is B two octaves higher (990 cycles/sec). The sound of the stopped string is B 495 because the ratio of the stopped string to the open string is 2:3. Therefore the wavelength of the stopped note is 2/3 that of the fundamental, and the frequency will be 1.5 times higher than the fundamental.

The harmonic at a ratio of 1:3 is a little different. The harmonic is two octaves higher, because waves that cannot pass through 5 evenly spaced nodes are dampened out. Thus the surviving tone is two octaves higher than the fundamental, or E 1320. The stopped string is A 440.

The ratios are, in order, 1:2, 2:3, 3:4, the first four counting numbers. They also sum to 10, which, as everyone knows, humans and other critters tend to have ten of this or ten of that. Pythagoreans attributed great mystical significance to these numbers and created a symbol from them, the *tetractys*. See Figure 4.

He extended these observations to strings that were weighted with different

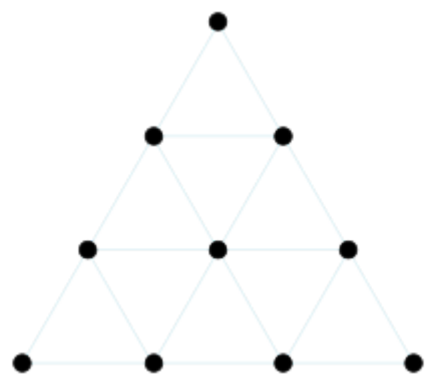


Figure 3: The Tetractys

weights. If twice the weight was suspended from the end of a string, the tone would be one octave higher. If 1.5 times the weight were attached, the tone would be that of the perfect fifth. He also experimented with bells and tuning forks of different weights and again found the same relationship: a bell with twice the weight of another would sound one octave lower, and so on. His investigations into harmonics and tones was quite complete, and so I credit him for being the first experimental physicist of Western history. Here he stands in clear contrast to the later Greek philosophers, especially Plato and Aristotle, who never checked their theoretical conclusions with empirical tests.

We can see how the Pythagoreans found meaning in numbers. These mystical side trips, while entertaining, did lead to an undermining of the belief that numbers held an inner harmony or insight that was the basis of all epistemology. How this happened is, again, obscured by the remoteness of time and the fog of legend. It was likely appreciated very early on that there is an audible contradiction lurking in the spare symmetry of the tetractys.

It is first of all, important to establish some fundamental facts. The orderings of the intervals are what we would call today exact solutions of the wave equation for the vibrating string. There can only be waves that fit two, then three, then four nodes and so on that satisfy the wave equation. So for tones within the octave, the ordering 1:2, 2:3, 3:4 is correct. To first order, Pythagoras' Wave Theory is correct. However, music even in Pythagoras time had a much wider dynamic range than one octave.

Pythagoras went a step farther by predicting that by cycling through the scale a perfect fifth interval at a time, one could map out seven octaves of pitch. In Table II is shown the Pythagorean tuning based on C 64, two octaves below middle C (256 cycles/sec.)

The high C that is the result of perfect fifths tuning is not exactly seven octaves higher than C 64. That pitch would be

$$64 \cdot 2^7 = 8192 \text{ cycles/sec.}$$

On the other hand, skipping by fifths results in

$$64 \cdot \left(\frac{3}{2}\right)^{12} = 8303.77 \text{ cycles/sec.}$$

This discrepancy, 112 cycles/sec. in round numbers, is huge and amounts to almost one half of a chromatic interval at C 8192. It is known as the *wolf interval.* To get around this obstacle, Pythagoreans ignored C 64 and used only the intervals G through high C. This resulted in a badly out-of-tune scale, meaning that any music that combines these two intervals of C is unplayable. Upon reflection it is easy to see that there is no integer n that solves

$$\left(\frac{3}{2}\right)^n = 2^7$$

We cannot fault Pythagoras for this oversight. The manipulation of exponential equations was rudimentary at best in 400 BC. The first, and probably only prediction of Pythagoras wave theory fails. More interesting is why it fails:

> The mathematical tools necessary for its understanding had not yet been invented.

However, in fairness to Pythagoras, we must realize that a seven octave dynamic range is about one octave more than contemporary concert instruments, and about three octaves more than instruments of Pythagoras' day. A cathedral organ has a dynamic range of about six octaves. Orchestral strings have a practical range that covers about seven and one-half octaves, so the wolf interval might be heard on occasion with orchestral strings. On the other hand, instruments of Pythagoras' time such as the lute and the harp, were probably limited to three or four octaves.

Table II shows the estimated error in tuning by perfect fifths. It is based on a realistic low pitch of C that is two octaves below middle C. As we can see from Table II, the error in Pythagorean tuning was apparent at two or three octaves. So musicians of his day must have been aware that tuning by perfect fifths didn't really work. They would still have to tune their instruments by tempering the scale to reduce dissonance to a minimum.

This failure may have sown the seeds of doubt in the minds of some of Pythagoras followers who were musicians. Those with perfect pitch would have noticed a dissonance building after only the first octave. At large, Pythagoras lived the life of someone semi-divine. Men living to such perfection may be allowed one mistake, but two?

The inhabitants of Croton solved their problem with sword and fire. The solution to the wolf interval is the tempered scale. There are many flavors of tempering, and different schools of music prefer different ones. A simple one that illustrates tempering is one in which the perfect fifth is replaced with an interval that is just a little flat, but that eliminates the wolf interval. With the use of logarithms[4], we find the ratio of intervals is 1.4983 instead of 1.5. This results in a low G with pitch of 95.89 cycles/sec. instead of 96 cycles/sec. The difference in these two frequencies, when played together, is only about 0.11 cycles/sec. This would be barely noticed by a musically sensitive ear. Many people could not hear it at all.

[4]John Napier invented the logarithm in 1614. Joost Bürgi independently invented the logarithm but published six years after Napier.

Table 2: Cycle of Perfect Fifths on C, Two Octaves Below Middle C; All units in cycles per second.

Perfect Fifth Intervals	Pythagorean Pitch		Pitch Estimated from Pythagorean Tuning	True Pitch	Difference Between Estimated Tone and True Tone	Estimated Chromatic Interval
C	64	64	64	64	0	4.57
G	96					
		C	130.29	128	2.29	6.86
D	144					
A	216					
		Middle C	**262.28**	**256**	**6.28**	**15.42**
E	324					
B	486					
		C	520.71	512	8.71	34.71
F♯	729					
		C	1041.43	1024	17.43	52.07
C♯	1093.5					
G♯ − *A*♭	1640.25					
		C	2108.89	2048	60.89	117.16
E♭	2435.50					
B♭	3690.56					
		C	4218	4096	122	264
F	5535.8					
C	8303.8	8192	8303.8	8192	111.8	395.4

Pythagoras later developed a cosmology based on the Music of the Spheres. The spheres were the regions in which the moon, the sun and the planets moved. All of these spheres were centered about the Earth, which was stationary. Within these spheres, the planets would move somewhat erratically, especially Mars, for which the orbit is very elliptical. Thus Mars is liable to fall behind then suddenly move forward. The same is true of the inner planets Mercury and Venus that speed around the Sun and never stray far from it when viewed from Earth.

Figure 4 is a depiction of the Pythagorean Monochord. It was also called the Mundane Monochord. Mundane here is translated as music of the spheres. A Pythagorean Monochord is nothing more than a string strung between two bridges. Pythagoras used it to investigate the harmonics of the string. The figure may be viewed as an inverted guitar or violin string. Also included in the figure are the four elements, earth (*Terra*), water (*Aqua*), air (*Aer*), and fire (*Ignie*). The symbol Γ denotes the earth, and from Γ to A is the region of water[5]. From A to B is the region of air, and so on. From C to D is the sphere of the moon, ☾. The planets are listed in ascending order, with the sun being a planet: mercury, ☿, venus, ♀, sun, ⊙. The sun is located midway, at the octave. The outer planets follow, with Mars, ♂, Jupiter, ♃, Saturn, ♄, and the firmament at the double octave.

The positions of the planets were thought to be related to the velocity at which the planets moved through the ether. The firmament moved the fastest so it was assigned to the double octave. The sun was thought to be midway between the Earth and the firmament, so it was assigned to the octave. The outer planets and the firmament were thought to move faster and emit a higher tone. The heliocentric universe and Newtonian physics changed all that. Today we know that the relationship of speed vs. orbit is just the reverse: the larger the radius of the orbit, the slower the planet moves; the sun and the firmament are essentially stationary with respect to each other.

When one attempts to calculate the speeds of the planets, one comes to different conclusions than those shown in the figure. Take the sun-firmament comparison. For every 365 revolutions the sun makes through the heavens, the firmament makes 366. Thus the sun loses one revolution per year to the circling firmament. How does this translate to a one octave pitch for the sun and a double octave pitch for the firmament? The Moon's motion under this cosmology is also puzzling. The Moon loses 12 revolutions per year versus the Sun. So why is it assigned to a pitch of C? Should it not be assigned a pitch of A?

[5]From Fludd's *De Musica Mundana*

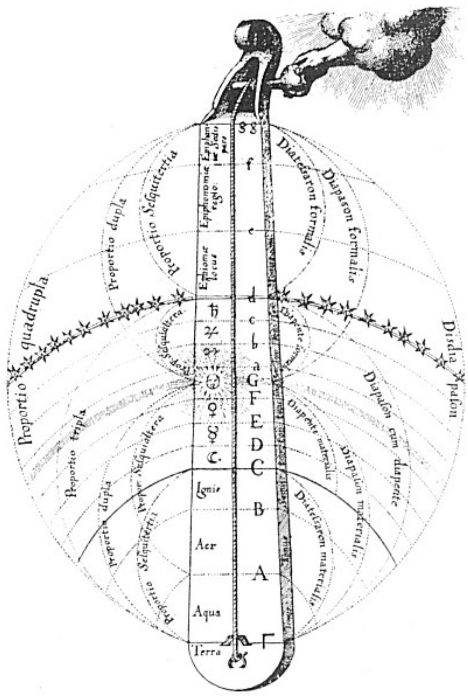


Figure 4: The Mundane MonoChord with its Proportions and Intervals.

5 Concluding Remarks

We can be grateful to Pythagoras for his mathematical investigations, and the Pythagorean Theorem. The invention of irrational numbers was an important breakthrough for early mathematics. His investigations of string harmonics led to an early understanding of wave phenomena that has become part of the foundation of modern physical theories. His philosophy of inner calm, living simply and soberly within a larger cosmic harmony, has undoubtedly helped many a troubled soul find peace and redemption.

Without intending him any disrespect, we can learn from his failures as well as from his successes. Though in modern times it seems that we live with the cult of the celebrity more than ever, there now exists a clear separation between science and mysticism. Big science more than ever depends on the contributions of many individuals, not just the superstars. Often there are dozens of authors for important theoretical and experimental papers, especially in physics. If Pythagoras is guilty of anything, it is that he was simply human, and extrapolated his initial brilliant insights to a false conclusion. In this way, however, he exemplifies the self-correcting nature of science. Rather quickly, he came to accept that irrational numbers existed, and may have been the one to devise the proof. As a musician, he undoubtedly found his perfect fifth tuning to be lacking. The harmonics of the string were correct and still intrigued.

His Music of the Spheres was eventually discredited, but it was a plausible first attempt at a Theory of Everything. As such it stands as the direct antecedent of modern theories of everything, including the Big Bang Theory.

Pythagoras' ability to ask questions outran his ability to answer them. And therein lies the humble conclusion of this cautionary tale: We never know just when our scientific questions have outrun our ability to describe them with mathematics or isolate them in the laboratory. The mathematics and physical experiments may not have been invented yet.

6 Aknowledgements

I am grateful for the many helpful comments of the audience at the meeting of the Philosophical Club of Cleveland. Their feedback contributed to the revision, and I believe improvement, of this paper.